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ALL SHORTEST ROUTES FROM A FIXED ORIGIN IN A GRAPH

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A shortest route is sought between a fixed origin node $i = 0$ to n other nodes in a graph when directed arc distances c_{ij} are given and the values of c_{ij} may be positive, negative, or zero $i \neq j$. No values c_{ij} are specified unless there is an arc from i to j. This problem (as is well known) includes the travelling salesman problem with distances $d_{ij} > 0$ because one can set $[c_{ij} = d_{ij} - K]$ where $K > \Sigma_i \Sigma_j d_{ij}$ and look for a minimum route from 0 back to itself. Therefore our objective will be more modest: To find a negative cycle in a graph if one exists or if none exists then to find all the shortest paths from the origin.

The method is inductive. On step k, there is a set S_k consisting of the origin and $k - 1$ other nodes. Restricting arcs to those that belong to the subgraph of S_k, the minimum distances from the origin along these arcs to nodes $i \in S_k$ are assumed known and have value Π_i. It is also assumed that no negative cycles exist in the subgraph of S_k. It follows that

$$(1) \quad \Pi_i + c_{ij} \geq \Pi_j \qquad \text{for all } i \in S_k, \; j \in S_k.$$

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Theorem 1: Let D_{ij} denote the length of the shortest route from i to j along arcs of the subgraph of S_k containing no negative cycles and let (1) hold, then

$$(2) \quad D_{ij} \geq \Pi_j - \Pi_i$$

Proof: Let the sequence $(i \; ; i_1 \, , i_2, \ldots, i_\lambda \; ; j)$ denote the nodes along a minimum route from i to j in S_k , then by (1) ,

$$\Pi_i + c_{ii_1} \geq \Pi_{i_1}, \; \Pi_{i_1} + c_{i_1 i_2} \geq \Pi_{i_2}, \ldots\ldots \; , \; \Pi_{i_\lambda} + c_{i_\lambda j} \geq \Pi_j \; .$$

Adding these inequalities together yields the desired relation.

Assuming now that we know the minimal distances Π_i for S_k , we wish to augment S_k by including a node $q \notin S_k$. We denote $S_{k+1} = \{S_k \, , \, q\}$ and wish to determine minimal distances Π_i^* from the origin along arcs of the subgraph of S_{k+1} to nodes $i \in S_{k+1}$. The theorem below permits us to determine Π_q^* immediately.

Theorem 2: Let $q \notin S_k$, and $S_{k+1} = \{S_k \, , \, q\}$ then a shortest route from 0 to q in S_{k+1} has as last arc of the route $(p \, , \, q)$ where $p \in S_k$ satisfies

$$(3) \quad \Pi_p + c_{pq} = \underset{i \in S_k}{\mathrm{Min}} \, (\Pi_i + c_{iq})$$

and $\Pi_q^* = \Pi_p + c_{pq}$ is the minimum distance from the origin to q in S_{k+1} .

Proof: Suppose false and a shorter route is via $\bar{p} \in S_k$, then

$$\Pi_{\bar{p}} + c_{\bar{p}q} < \Pi_p + c_{pq}$$

contradicting (3) . This theorem is true even if S_k has negative cycles. The Π^*_q and Π_i would then represent the shortest distance without cycles from the origin.

Knowing Π^*_q , Theorem (4) below may now be applied to determine for another node $\ell \in S_{k+1}$, its minimum distance Π^*_ℓ from the origin along arcs of the subgraph of S_{k+1} . Knowing Π^*_q and Π^*_ℓ we reapply Theorem (4) again and again, each time finding a least distance for another node in S_{k+1} . This is done until all nodes are exhausted in S_{k+1} or the optimality condition $\delta_{ij} \geq 0$ of Theorem 3 below is satisfied in which case the remaining Π_i values are also optimal for S_{k+1} , or the negative cycle condition of Theorem 5 is satisfied.

Theorem 3: Let T be any subset of nodes i whose minimum distance Π^*_i from the origin along routes in the subgraph of S_{k+1} is known, let $q \in T$; let S_k and T contain no negative cycles; let

$$(4) \qquad \delta_{ij} = \Pi^*_i + c_{ij} - \Pi_j \qquad i \in T \, , \; j \notin T$$

then, if

$$(5) \qquad \delta_{ij} \geq 0 \qquad \text{for all } i \in T \, , \; j \notin T$$

the minimum distance for all remaining nodes is

$$(6) \qquad \Pi^*_j = \Pi_j \qquad \text{for all } j \notin T$$

This theorem is true even if T contains negative cycles but requires a different proof.

<u>Proof</u>: The conditions for optimality in S_{k+1} analogous to (1) are:

$$\text{(7)}\qquad \begin{array}{llll} \delta_{ij} = \Pi^*_i + c_{ij} - \Pi_j \geq 0 & & i \in T, & j \notin T \\ \Pi_i + c_{ij} - \Pi_j \geq 0 & & i \notin T, & j \notin T \\ \Pi^*_i + c_{ij} - \Pi^*_j \geq 0 & & i \in T, & j \in T \\ \Pi_i + c_{ij} - \Pi^*_j \geq 0 & & i \notin T, & j \in T \end{array}$$

The first of these holds by hypothesis (5), the second by (1), the third by hypothesis that the T set is optimal in S_{k+1} (and there are no negative cycles in T); finally the fourth because $\Pi^*_j \leq \Pi_j$ and (1) holds.

On the other hand if the optimality conditions $\delta_{ij} \geq 0$ of Theorem 3 does not hold for all $i \in T$, $j \notin T$, then $\delta_{t\ell} = \text{Min}\ \delta_{ij} < 0$ holds for some $t \in T$ and $\ell \notin T$. It will be shown in Theorem 4, that the minimum distance from the origin along arcs of the subgraph of S_{k+1} to node ℓ is given by $\Pi^*_\ell = \Pi_\ell + \delta_{t\ell}$. Thus Theorem 4 may be reapplied until there are no longer any nodes in S_{k+1} not in T or condition (5) holds, or a negative cycle is detected, but we will speak more about this later in Theorem 5.

<u>Theorem 4</u>: Let S_k and T contain no negative cycles where T is any subset of nodes i whose minimum distances from the origin in

S_{k+1} is Π_i^*. If for some $t \in T$, $\ell \notin T$

(8) $\quad \delta_t = \mathrm{Min}\ \delta_{ij} < 0 \qquad i \in T,\ j \notin T$

then

(9) $\quad \Pi_\ell^* = \Pi_\ell + \delta_{t\ell} = \Pi_t^* + c_{t\ell}$

<u>is the minimal distance from the origin along arcs in the subgraph of S_{k+1} to node ℓ.</u>[1)]

<u>Proof</u>: On the contrary, if there is a shorter route to ℓ, then this route must include the node q and perhaps some other nodes of T (otherwise Π_ℓ would be minimum but we know $\Pi_\ell^* < \Pi_\ell$ by (8) and (9). Along this shorter route let $(\bar{t}, \bar{\ell})$ be the last arc such that $\bar{t} \in T$, $\bar{\ell} \notin T$. Then the distance along the route from $\bar{\ell}$ to ℓ, may be denoted by $D_{\bar{\ell}\ell}$ (see Theorem 1) because the nodes from $\bar{\ell}$ to ℓ are all elements of S_k. By Theorem (1)

(10) $\quad D_{\bar{\ell}\ell} \geq \Pi_\ell - \Pi_{\bar{\ell}}$

On the other hand by virtue of the assumed shorter route through $\bar{t}$, $\bar{\ell}$

(11) $\quad \Pi_{\bar{t}}^* + c_{\bar{t}\bar{\ell}} + D_{\bar{\ell}\ell} < \Pi_t^* + c_{t\ell}$

---

1) This theorem also holds if T contains negative cycles and Π_i^* are the shortest distances from the origin along routes without cycles.

Subtracting (10) from (11) and rearranging

$$\Pi_{\bar{t}}^{*} + c_{\bar{t}\ell} - \Pi_{\ell} < \Pi_{t}^{*} + c_{t\ell} - \Pi_{\ell}$$

or $\delta_{\bar{t}\ell} < \delta_{t\ell}$ by (4) which contradicts hypothesis (8) of Theorem 4.

<u>Theorem 5</u>: <u>If</u> S_k, T <u>contain no negative cycles and the shortest distance from the origin in</u> S_{k+1} for $i \in T$ is $\Pi_i^* < \Pi_i$ and T is <u>augmented to</u> $T^* = \{T, \ell\}$ <u>where ℓ is as defined in Theorem 4, then a a necessary and sufficient condition that</u> T^* <u>contain a negative cycle is</u>

$$(12) \qquad \Pi_{\ell}^{*} + c_{\ell q} - \Pi_{q}^{*} \quad \delta_{\ell q} < 0$$

<u>Proof</u>: Since $\Pi_i^* < \Pi_i$ holds the optimal route from the origin to ℓ in S_{k+1} passes through q. If (12) holds, then the cycle consisting of the optimal route from q to ℓ and then arc (ℓ, q) has negative length. This may be seen by summing the relations $\Pi_i^* + c_{ij} = \Pi_j^*$ along the route from q to ℓ and then adding it to (12). If, on the other hand, (12) does not hold, then we will show that $\Pi_i^* + c_{ij} \geq \Pi_j^*$ for all $i \in T^*$, $j \in T^*$ which implies that no negative cycle in T^* exists (as one can see by summing such relations over the arcs of a cycle.)

We need now only rule out for some i_o and $j_o \neq q$ that $\Pi_{i_o}^* + c_{i_o j_o} < \Pi_{j_o}^*$. This would mean we could lower the value of $\Pi_{j_o}^*$ by making i_o the node that precedes j_o along the optimal route instead of some i_1. This <u>deletion</u> of the arc $(i_1\, j_o)$ from

the tree[2] of optimal routes and entering the arc $(i_\circ j_\circ)$ into the tree either would provide a shorter route to $j_\circ$ or it would cause a cycle to form which (by an earlier argument) is negative. However neither is possible because the former implies a shorter route to $j_\circ$ (because $\Pi^*_{j_\circ}$ was lowered) while the latter implies a negative cycle <u>not</u> involving q . The cycle cannot involve q because all shortest routes $i \in T^*$ from the origin pass through q and there are no directed arcs into q along the tree of optimal routes in T^* . But a negative cycle in S_k is contrary to assumption.

<u>Thus a negative cycle will always be found if there is one by (12). If one is found the inductive process terminates.</u>

The following theorem due to M. Sakarovitch (verbal communication) permits one to find the minimal distance in S_{k+1} to several nodes at once.

<u>Theorem 6 (Sakarovitch): Let</u> L <u>be the nodes in the tree of optimal routes in</u> S_k <u>which are successors[3] of</u> ℓ <u>as defined in Theorem 4, then</u>

$$(13) \qquad \Pi^*_i = \Pi_i + \delta_{t\ell} \quad \text{for } i \in L .$$

---

[2]Note: If there are no negative cycles in S_k and T in S_{k+1} there is a tree of optimal routes to $i \in T$ branching out from the origin; also the added arc (t,ℓ) with $t \in T, \ell \notin T$ still yields a tree of shortest routes without cycles in $i \in T^*$.

[3]The tree of optimal routes from the origin forms a partially ordered set. The "successors" of ℓ are those nodes reached through ℓ .

Proof: One notes first that the distance $\Pi_i + \delta_{t\ell}$ can be realized by first going along the optimal route to ℓ and then along the former route from ℓ to $i \in L$. Now assume on the contrary that there is a better route to i. As in proof of Theorem 4, let $\bar{t}\bar{\ell}$ be the last arc of a better route such that $\bar{t} \in T$ and $\bar{\ell} \notin T$, then $\Pi^*_{\bar{t}} + c_{\bar{t}\bar{\ell}} + D_{\bar{\ell}i} < \Pi_i + \delta_{t\ell}$. Subtracting $D_{\bar{\ell}i} \geq \Pi_{i\ell} - \Pi_{\bar{\ell}}$, yields $\delta_{\bar{t}\bar{\ell}} < \delta_{t\ell}$ contrary to (8).

For completeness we give the following well known theorem, [3].

Theorem 7: If $c_{ij} \geq 0$ and Π_i of S_k are known to be the minimal distances from the origin for the k nodes of S_k using arcs of the full n-node problem, then $\Pi_q = \Pi_p + c_{pq}$ is the minimal distance for $q \notin S_k$ where

$$\Pi_p + c_{pq} = \min_{\substack{i \in S_k \\ j \notin S_k}} (\Pi_i + c_{ij}) \quad , \; p \in S_k \tag{14}$$

Proof: If not, then q is reached via some shorter route that has nodes in common with S_k (since S_k includes the origin). Let $(\bar{t}, \bar{q})$ be the last arc on the shorter route with $\bar{t} \in S_k$ and $\bar{q} \notin S_k$, then

$$\Pi_{\bar{t}} + c_{\bar{t}\bar{q}} + (\text{min distance } \bar{q} \text{ to } q) < \Pi_p + c_{pq} \tag{15}$$

but this relation contradicts (14) because minimum distance from $\bar{q}$ to q is non-negative when $c_{ij} \geq 0$.

We are now in a position to give a count on the number of

additions. Associated with each set of additions such as for (14) is the same number of comparisons (or possibly one less). In the case $c_{ij} \geq 0$, the same sums occur in S_k and S_{k+1} for the same (i, j). Since at step $k+1$ we do not need to consider the arcs back to S_k, the total additions do not exceed the total number of arcs. We will denote this total by A . The procedure is to sort the $\Pi_i + c_{ij}$ values as generated from low to high. Let the lowest sum on this list be $\Pi_i + c_{ij_\bullet}$. This sum on the list is deleted if $\Pi_{j_\bullet}$ has previously been determined; if not then $\Pi_{j_\bullet} = \Pi_i + c_{ij_\bullet}$. Next the sums $\Pi_{j_\bullet} + c_{j_\bullet k}$ are computed for all arcs (j, k) and made part of the sorted list. The process is then repeated. Sorting requires effort, however, and so that the two theorems that follow are misleading.

Theorem 8: If all distances $c_{ij} \geq 0$, then the number of additions using formula (14) does not exceed A , the number of arcs.

Theorem 9: The number of additions in the general case, when formula (3) and (8) is used does not exceed

$$(16) \qquad A + nf_1 + (n - 1) f_2 + \ldots f_n$$

where n is the number of nodes, f_k is number of arcs directed forward from the k-th node to enter the induction.

This suggests preordering from low to high the nodes by the number of their forward arcs. If this is done, the bound reduces to

$$(17) \qquad A + nf_1 + (n - 1) f_2 + \ldots f_n \leq (n + 3) A/2$$

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A shortest route is sought between a fixed origin to other nodes in a graph when directed arc distances are given which may be positive, negative, or zero. This problem as stated includes the difficult travelling salesman problem. A less difficult problem is considered instead, namely: To find a negative cycle in a graph if one exists or if none exists then to find all the shortest paths from the origin. The method is inductive on nodes.

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